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Quantum Mechanics and Geometric Analysis on Manifolds[1]

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A central idea of modern geometric analysis is the assignment of a geometric structure, usually called the *symbol*, to a differential operator. It is known that this operation is closely related to quantum mechanics. For a class of linear operators, including the Dirac operator, a geometric structure, called a *co-Riemannian metric*, is assigned to such symbols. Certain other topics related to the geometric structure of quantum mechanics, e.g., the symplectic structure of the projective space of Hilbert space, are briefly treated.

1. INTRODUCTION

From the perspective of contemporary mathematics, classical mathematical physics is a mixture of material from what we now call "calculus on manifolds," Lie group theory, and "concrete" functional analysis (to use the marvelous phrase of Paul Lévy, 1924). Quantum mechanics, as developed in the 1920s, grew from these roots to involve a more abstract level of mathematics. All contemporary treatises follow the lead of von Neumann (1955) and emphasize this "abstract" functional analysis. This creates a discontinuity between quantum mechanics and other physical theories that might be in part responsible for the difficulty in making a mathematically viable and useful relation between quantum field theory and elementary particle theory.

I have long thought there might be an alternative foundational treatment of quantum mechanics in terms of ideas of manifold differential

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geometry. I have already published several fragments of my ideas (Hermann, 1965; 1966; 1970a,b; 1972a; 1973a–d; 1974; 1975; 1977a,b; 1978), and hope it might be fruitful to develop a fuller sketch of a possible theory. I emphasize the word *sketch*—much foundational work remains before it is a serious competitor to the functional analysis viewpoint.

The essence of a "geometric" approach to analysis is to attach "geometric structures" to analytic and algebraic objects, to find a "space" where abstract objects may *live*. While conventional quantum mechanics starts off with a Hilbert space (or an alternate algebro-analytic structure, like a C^* algebra) as *given* axiomatically, a geometric approach will emphasize that the objects are "tied down" geometrically, e.g., as cross sections of fiber bundles on manifolds. I will try to develop such ideas here for the foundations of quantum mechanics. I will not work in maximum generality, but will concentrate on the geometric description of the Hilbert spaces associated with Dirac-like equations and certain associated "infinite-dimensional symplectic manifolds." First, I will review some aspects of the theory of two relevant mathematical structures, the first-order linear differential operators (like the Dirac operator) and the Riemannian metrics and cometrics (Hermann, 1973c,d; 1976).

Such material involves the traditional sort of "geometric objects" attached to *finite-dimensional* manifolds. There are also vast possibilities of creating a *new* differential geometry of *infinite-dimensional* spaces, much in the spirit of Lévy's (1924) (and Volterra's) pioneering efforts. At the end of this review, I will briefly go into one such effort, an outgrowth of Hermann (1965). (The "quantum phase space" is the *cotangent bundle* of the space of smooth probability measures on the *classical* configuration space.) There are obvious possibilities of similar theories to describe quantum and field-theoretic phenomena. Until recently, the mathematical theory of infinite-dimensional manifolds and differential geometry has not been well adapted to the needs of the physical and system-theoretic applications. However, the recent hybrid theory of *pseudodifferential and Fourier integral operators* (Treves, 1980; Taylor, 1981; Grossman et al., 1968) is much better adapted to these needs, and I am more optimistic that there is some payoff to the effort of introducing more general machinery. Of course, this subject itself has part of its roots in mathematical physics and quantum mechanics, particularly the WKB and Feynman path integral expansions.

Another subject of great potential for application in physics and engineering is the theory of the sort of infinite-dimensional Lie groups and algebras called *gauge groups* and *current algebras* (Hermann, 1966; 1969; 1970a; 1973b). (In the recent mathematical literature, they are called *local Lie algebras*.) These objects are also closely linked to a theory of infinite-dimensional manifolds.

It is an honor for me to dedicate this material to Professor Dirac, since it is he who has initiated and developed this "geometric" picture of quantum mechanics. Perhaps one might hope that, at this geometric level, there will someday be a Hegelian synthesis of the ideas of Einstein and other founders of quantum mechanics.

I would also like to thank Professor Gerhard Emsch, who made useful suggestions about improving a first draft of this work.

2. FIRST-ORDER LINEAR DIFFERENTIAL OPERATORS ON VECTOR BUNDLES OVER MANIFOLDS

Let X be a manifold. (Assume for simplicity that all manifolds are finite dimensional and C^∞. Maps between manifolds are "smooth," i.e., C^∞, unless mentioned otherwise.) Let $\mathfrak{F}(X)$ be the algebra of C^∞ real-valued functions over X. Let

$$\pi: E \to X$$

be a vector bundle over X whose fibers are finite-dimensional, complex vector spaces. Let $\Gamma(E)$ be the space of cross sections, considered as an $\mathfrak{F}(X)$ module.

For each integer m, let $D^m(E)$ be the space of mth-order, linear differential operators: $\Gamma(E) \to \Gamma(E)$. What is characteristic of "modern" (i.e., post-1960) differential geometry is to attach *geometric* structures to such differential operators. Previously, it was quite common to attach differential operators to geometric structures. The Laplace–Beltrami operator attached to a Riemannian metric was the traditional example. Another such theory was the main theme of Darboux' monumental treatise *La Theorie des Surfaces*: studying surfaces by attaching second-order linear differential operators in two independent variables to them. Many ideas that have proved so fruitful in modern inverse scattering-soliton theory can also be discerned in that treatise, e.g., the Bäcklund transformation for the sine-Gordon equation.

In this paper, I shall concentrate on $m = 1$, i.e., on attaching geometric structures to first-order, linear differential operators. (The Dirac equation is, of course, the prototype.) Now, what we have learned in the theory of "pseudodifferential operators" (Treves, 1980; Taylor, 1981; Hermann, 1977c) is the importance of attaching a geometric object called a *symbol* to differential operators. This symbol will (for linear differential operators) be a cross section of a vector bundle. In Hermann (1973a) I have developed a way of doing this which is a generalization of the way vector fields are defined in differential operators, i.e., as derivatives of rings of functions. Let us review this now for the easiest use of a first-order linear differential operator.

Let

$$D\colon \Gamma(E) \to \Gamma(E)$$

be such an operator. For $f \in \mathfrak{F}(X)$, let

$$\sigma(f, D)$$

be the commutator of D with multiplication by f:

$$\sigma(f, D)(\gamma) = D(f\gamma) - fD(\gamma) \qquad \text{for } \gamma \in \Gamma(E) \tag{1}$$

By the very definition (Hermann, 1973a) of "first-order operator," $\sigma(f, D)$ is a zeroth-order operator, i.e., is an $\mathfrak{F}(X)$ linear mapping of $\Gamma(E) \to \Gamma(E)$. The following algebraic property is the key to assigning a "geometric" structure to the operator.

Theorem 2.1. If $D \in D^1(E)$, then

$$\sigma(f_1 f_2, D) = f_1 \sigma(f_2, D) + f_2 \sigma(f_1, D) \qquad \text{for } f_1, f_2 \in \mathfrak{F}(X) \tag{2}$$

Proof. Let m_f be the operator of multiplication by $f \in \mathfrak{F}(X)$. Then, by definition

$$\sigma(f, D) = [D, m_f] \tag{3}$$

Hence,

$$\begin{aligned} \sigma(f_1 f_2, D) &= [D, m_{f_1 f_2}] \\ &= [D, m_{f_1} m_{f_2}] \\ &= [D, m_{f_1}] m_{f_2} + m_{f_1} [D, m_{f_2}] \end{aligned}$$

By hypothesis, $[D, m_f]$ is a zeroth-order operator, i.e., it commutes with m_{f_2}. Hence,

$$\sigma(f_1 f_2, D) = m_{f_2} [D, m_{f_1}] + m_{f_1} [D, m_{f_2}]$$

which, after using (3), gives relation (2). ■

Algebraically, relation (2) says that σ is a derivation of the algebra $\mathfrak{F}(X)$ into the algebra of all linear operators on $\Gamma(E)$. In case E is the product $X \times R$, i.e., $\Gamma(E)$ is $\mathfrak{F}(X)$ itself, (2) is the familiar algebraic identity characterizing vector fields on X, i.e., cross section of the tangent bundle $T(M)$. Thus, in this case the map

$$f \to \sigma(f, D)$$

is a vector field, i.e., an element of $\Gamma(T(M))$. Let $\sigma(D)$ denote this vector field. Call it the *symbol* of D. It is known (Hermann, 1977c) that every such vector field arises from a derivation of $\mathfrak{F}(X)$. We have then proved (as a warm-up for the general case of Dirac-type equations) the following result:

Theorem 2.2. Suppose $E = X \times R$ so that $\Gamma(E) = \mathfrak{F}(X)$. The symbol map

$$D \to \sigma(D)$$

maps $D^1(E)$ linearly onto $\Gamma(T(X))$, the vector fields on X. It is a Lie algebra homomorphism [with $D^1(E)$ made into a Lie algebra under commutator]. This map exhibits $D^1(E)$ as a semidirect sum of the Lie subalgebra $\Gamma(T(X))$ and the Lie ideal $D^0(E)$.

Now let us go to the case of a general vector bundle E. Let E^d be the dual bundle, i.e., the fiber over a point $x \in X$ is the dual space to the vector space $E(x)$, which is the fiber of E. Then, for $\gamma \in \Gamma(E)$, $\theta \in \Gamma(E^d)$, the map

$$f \to \theta(\sigma(f, D)\gamma)$$

is a derivation of $\mathfrak{F}(X)$ into itself, i.e., a vector field. This defines a *bilinear* map

$$\sigma(D) \colon \Gamma(E) \times \Gamma(E^d) \to \Gamma(T(X))$$

Note that this is $\mathfrak{F}(X)$-bilinear, i.e., commutes with multiplication by functions. This property is the algebraic equivalent of what, geometrically, is defined as a cross section of a vector bundle. The vector bundle in this case is $T(E) \otimes E^d \otimes E$. Dually, $\sigma(D)$ can also be considered as a *linear* map

$$\Gamma(T^d(X)) \to \Gamma(E) \otimes \Gamma(E^d)$$

Now, the cross sections of $T^d(X)$ are the one-different forms on X. Let θ be such a form. Then, the symbol operation assigns to θ a cross section

$\sigma(D)(\theta)$ of the bundle $E \otimes E^d$, i.e., for each $x \in X$:

$$\sigma(D)(\theta)(x)$$

is a linear map: $E \to E$.

Let us sum up as follows.

Theorem 2.3. Let D: $\Gamma(E) \to \Gamma(E)$ be a first-order linear differential operator on cross sections of the vector bundle E. The symbol $\sigma(D)$ is a cross section of the tensor product of the tangent bundle $T(X)$ to X and the bundle $E \otimes E^d$ whose fiber at each point $x \in X$ is the space of linear maps: $E(x) \to E(x)$. $\sigma(D) = 0$ if and only if D is a zeroth-order differential operator, i.e., a cross section of $E \otimes E^d$.

All this takes a simple form in a more traditional notation using coordinates and a local product structure for E. Suppose

$$(x^\mu), \qquad 0 \leqslant \mu \leqslant n-1$$

is a coordinate system of functions on X. Let

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

be corresponding dual basis of vector fields. Locally, we have

$$E \simeq X \times V$$

where V is a vector space. $\Gamma(E)$ can then be identified with the space of maps γ: $X \to V$. A first-order differential operator D: $\Gamma(E) \to \Gamma(E)$ is then of the form

$$D = A^\mu \partial_\mu + A \tag{4}$$

where $x \to A^\mu(x)$, $A(x)$ are maps from X to $L(V, V)$, the vector space of linear maps: $V \to V$ (or $m \times m$ matrices, if $V = \mathbb{C}^m$). Then,

$$\sigma(D, f) = A^\mu \partial_\mu(f) \qquad \text{for } f \in \mathfrak{F}(X) \tag{5}$$

Also,

$$\sigma(D, f) = A^\mu df(\partial_\mu)$$

hence

$$\sigma(D)(\theta) = A^{\mu}\theta(\partial_{\mu}) \tag{6}$$

if θ is a one-differential form on X.

Thus, all this elaborate algebraic formalism is not really essential. However, it serves three useful purposes:

1. It proves, a priori, that the operator on the right-hand side of (6) is independent of coordinates for X and the local product structure for the vector bundle E.
2. It emphasizes the underlying nature of the assignment

   $$D \to \sigma(D)$$

   as a cross section of a vector bundle assigned to a "differential operator." It may be thought of as a passage from "analysis" to "geometry."
3. Its purely algebraic nature suggests extensions to spaces X which are broader than the class of finite-dimensional, differentiable manifolds. Two possibilities are the following:
   a. Manifolds with singularities
   b. Infinite-dimensional manifolds.

Although I will not go into it here, each of these extended classes of manifolds has algebras attached to it (in terms of "sheaves" or "schemes") that plays the same role in the more extended theories that the C^{∞} functions play in ordinary manifold theory.

3. CO-RIEMANNIAN METRICS

Let X be a manifold. In differential geometry, a *Riemannian metric* is a cross section of the symmetric tensor product of the cotangent bundle with itself:

$$\Gamma(T^{d}(X) \otimes T^{d}(X))$$

[In classical tensor language, it is a tensor field of type (0,2).] Alternatively, it is a symmetric $\mathfrak{F}(X)$-bilinear map:

$$g\colon \Gamma(T(X)) \times \Gamma(T(X)) \to \mathfrak{F}(X)$$

Such an object is said to be *nonsingular* if it is nondegenerate, i.e., if

$$g(\Gamma(T(X)), V) = 0 \qquad \text{for } V \in \Gamma(T(X)) \qquad \text{implies } V = 0$$

Of course, when one speaks (in either mathematics or physics) of a "Riemannian metric," one usually only means a nonsingular one.

About ten years ago, I started development of the theory of the dual objects that I call co-Riemannian metrics (Hermann, 1973b–d; 1975; 1976). As we shall see, they appear naturally in terms of the geometric structure of linear differential operators (and quantum mechanics!).

Definition. A co-Riemannian metric on X is a cross section g^d of the symmetric tensor product bundle

$$T(X) \circ T(X)$$

i.e., a $\mathfrak{F}(X)$-bilinear symmetric map

$$g^d\colon \Gamma(T^d(X)) \otimes \Gamma(T^d(X)) \to \mathfrak{F}(X)$$

which assigns a function $g^d(\theta_1, \theta_2)$ to each pair (θ_1, θ_2) of one-differential forms on X. Such a g^d is said to be *nonsingular* if the following condition is satisfied:

$$g^d(\Gamma(T^d(X)), \theta) = 0 \qquad \text{implies } \theta = 0$$

If g^d is nonsingular, the symmetric matrix defining the tensor field in local coordinates can be inverted, and g^d defines a Riemannian metric in the usual sense. However, the singular cases also occur naturally, and the "Riemannian metrics" and their "duals" should be studied separately. [A similar remark applies to the "symplectic structures", i.e., the "cosymplectic structure" should be studied as well. See Hermann (1975; 1977b; 1976). These structures do come up in many interesting physical situations, especially in Dirac's work.]

In Hermann (1975; 1973c,d; 1976) , some of the rudimentary notions associated to these "co-Riemannian metrics" were briefly developed (for example, the generalization of what is meant by "geodesic," and the ways this geometric notion can be applied in analysis), and I will not go into it here. One of the most important geometric features of the "co" objects is their behavior under mappings. Riemannian metrics "pull-back" under mapping, the co-Riemannians "push forward." This is closely related to the concept of "Riemannian submersion," which plays a role in many physical theories that use Riemannian geometric ideas.

An important use of the co-Riemannian metric is in connection with the role of *symmetry groups* of the underlying geometric structure. Let g^d be a co-Riemannian metric on X, and let G be a transformation group on X consisting of automorphisms of the structure. Suppose (for simplicity) that the action of G on X is *regular* in the sense that the orbit space $G\backslash X$ can be made into a manifold, with the quotient map ϕ: $X \to G\backslash X$ (assigning to each $x \in X$ the orbit on which it lies) being a submersion. Then, there is a co-Riemannian metric g'^d on $G\backslash X$ such that

$$\phi_*(g^d) = g'^d$$

Let us now return to the study of first-order linear differential operators, and the study of co-Riemannian metrics to which they give rise.

4. CO-RIEMANNIAN METRICS ASSOCIATED WITH FIRST-ORDER LINEAR DIFFERENTIAL OPERATORS

Let us now return to the situation of Section 2. $E \to X$ is a complex vector bundle over the manifold X. D: $\Gamma(E) \to \Gamma(E)$ is a first-order linear differential operator. Let $L(E)$ be the vector bundle over E whose fibers over each point $x \in E$ is the space $L(E(x), E(x))$ of linear maps: $E(x) \to E(x)$ of the fiber of E into itself. [In a formula, $L(E) = E \otimes E^d$.] The symbol of D can be considered as a map

$$(x, \theta) \to \sigma(D)(\theta)$$

which assigns an element of $L(E)(x)$ to each covector $\theta \in X_x^d$. Now, set

$$g^d(\theta_1, \theta_2) = \tfrac{1}{2}\operatorname{tr}(\sigma(D)(\theta_1)\sigma(D)(\theta_2)^*) \tag{7}$$

g^d is then a complex-valued *symmetric* bilinear form on X_x^d. As x varies, its real and imaginary parts define co-Riemannian metrics on X (in general, singular, of course).

Let us recall how this goes for the Dirac equation:

$$X = R^4$$

$$E = X \times \mathbb{C}^4$$

Space-time coordinates on X are denoted by (x^μ), $0 \leqslant \mu \leqslant 3$. Let

$$D = A^\mu \partial_\mu \tag{8}$$

where (A^μ) are (constant) 4×4 matrices such that

$$A^\mu A^\nu + A^\nu A^\mu = g^{\mu\nu} I \tag{9}$$

where $(g_{\mu\nu})$ [the inverse matrix to $(g^{\mu\nu})$] is the Lorentz metric tensor

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Apply the trace operator to both sides of (9):

$$\operatorname{tr}(A^\mu A^\nu) + \operatorname{tr}(A^\nu A^\mu) = 2g^{\mu\nu}$$

or

$$\operatorname{tr}(A^\mu A^\nu) = g^{\mu\nu} \tag{10}$$

Then,

$$\sigma(D)(\theta) = A^\mu \theta(\partial_\mu)$$

Hence,

$$\begin{aligned} g^d(\theta_1, \theta_2) &= \tfrac{1}{2}\operatorname{tr}\big(A^\mu \theta_1(\partial_\mu) A^\nu \theta_2(\partial_\nu)\big) \\ &= g^{\mu\nu} \theta_1(\partial_\mu)\theta_2(\partial_\nu) \qquad \text{using (10)} \end{aligned} \tag{11}$$

We have proved the following theorem:

> *Theorem 4.1.* The co-Riemannian metric associated with the Dirac equation is dual to the Lorentz metric.

Thus we see concretely how a "geometric" structure may be determined by an analytic one. One may readily go further and study Riemannian metrics more general than flat ones determined by first-order differential operators. In the physics literature this goes under the slogan "The Dirac equation in General Relativity." Deformation of the metric leads, via the method of Belinfante and Rosenfeld, to the *energy-momentum* tensor (Hermann, 1973d). However, further work in this direction will be pursued in another publication.

A further structure is needed in order that a linear differential operator define a *quantum system*, namely, a "conservation law" that defines a *probability structure*.

5. HILBERT SPACE STRUCTURES ON THE SPACE OF SOLUTIONS OF A FIRST-ORDER LINEAR DIFFERENTIAL OPERATOR

Let us return to the case of general complex vector bundle E over a manifold X with a first-order linear differential operator

$$D: \Gamma(E) \to \Gamma(E)$$

given. Let $\Gamma_0(E)$ be the space of cross sections with compact support, i.e., those that vanish outside of some compact subset of X. Since D is a differential operator, D maps $\Gamma_0(E)$ into itself. We will now investigate the possibility of defining positive definite Hermitian inner product structures (i.e., "incomplete Hilbert spaces")

$$(\gamma_1, \gamma_2) \to \langle \gamma_1 / \gamma_2 \rangle$$

on $\Gamma_0(E)$ by integration over submanifolds of X. Once this is done, these spaces can be completed with respect to the inner product to obtain complete Hilbert spaces, which can serve in the usual way as the raw material for quantum mechanics.

The simplest way to define Hermitian inner products on $\Gamma_0(E)$ is the direct generalization of the classical L_2-theory. For $x \in X$, k integer, let $\Lambda^k(X_x)$ be the space of k-multilinear, skew-symmetric, real-valued forms on X_x, i.e., the space of k-covectors of the tangent space. Let $\Lambda^k(X)$ be the corresponding vector bundle. The cross sections $\Gamma(\Lambda^k(X))$ are then the kth-degree differential forms on X.

Consider an R-bilinear bundle map

$$h: E \times E \to \Lambda^n(X) \otimes \mathbb{C}, \qquad n = \dim X$$

Thus, for $x \in X$, h maps the complex vector space fibers

$$E(x) \times E(x) \to \Lambda^n(X) \otimes \mathbb{C}$$

$$(e_1, e_2) \to h(e_1, e_2)$$

real-bilinearly. Suppose also that it is *Hermitian symmetric* and positive definite, i.e.,

$$h(e_1, e_2) = h(e_2, e_1)^*$$

$$h(e, e) > 0 \qquad \text{if } e \neq 0$$

$$h(e_1, ce_2) = ch(e_1, e_2) \qquad \text{for } c \in \mathbb{C}$$

Suppose now that $\gamma_1, \gamma_2 \in \Gamma_0(E)$. Then,

$$x \to h(\gamma_1(x), \gamma_2(x))$$

is an nth-degree differential form, which can be integrated over X. Set

$$\langle \gamma_1 / \gamma_2 \rangle = \int_X h(\gamma_1, \gamma_2) \tag{12}$$

Then, $\langle / \rangle$ satisfies all the conditions for a complex Hilbert space (except, of course, completeness).

Now, we must consider generalizations of (12). To allow for various interesting possibilities, all the data must be varied.

1. h takes values in $\Lambda^k(X) \otimes \mathbf{C}$, for $0 < k < n$.
2. h depends on the derivatives of γ_1, γ_2; i.e., h is a form defined on the jet-bundles $J^j(E)$ associated with E.
3. h depends on variables in addition to those of X; e.g., variables on a manifold that is a fiber space π: $Z \to X$ over X.

I will now describe one of the generalizations that seems to play a key role in the quantum mechanics of linear differential operators. Suppose Y is a manifold of dimension $n-1$. Let $G^{n-1}(X)$ be the Grassmann bundle over X, i.e., the fiber of $G^{n-1}(X)$ over a point $x \in X$ is the space of all $(n-1)$-dimensional linear subspaces of X_x. Thus, if

$$\phi \colon Y \to X$$

is a submanifold map, there is a map, which we denote by

$$\partial\phi$$

of $Y \to G^{n-1}(X)$, that assigns to each $y \in Y$ the $(n-1)$-dimensional, linear subspace

$$\phi_*(Y_y) = \partial\phi(y)$$

of $X_{\phi(y)}$.

Suppose now that h is a function of

$$y \in Y, \qquad \delta \in G^{n-1}(X)$$

$$e_1, e_2 \in E(x)$$

and takes values on

$$\Lambda^{n-1}(X)\otimes \mathbf{C}$$

Denote this map by

$$h(y, \delta, e_1, e_2)$$

Now, suppose that γ_1, γ_2 are elements of $\Gamma_0(E)$. The map

$$y \to h\big(y, \phi_*(y), \gamma_1(\phi(y)), \gamma_2(\phi(y))\big)$$

then defines an $(n-1)$st degree differential form on the $(n-1)$-dimensional manifold Y. The integral over Y is then defined as

$$\langle \gamma_1 / \gamma_2 \rangle$$

Of course, this inner product, as defined over *all* of $\Gamma_0(E)$, will not be *positive definite*. However, our goal is to define Hilbert space structures on the space $\Gamma(D)$ of solutions of a first-order linear differential operator

$$D\colon \Gamma(E) \to \Gamma(E)$$

One would also require that $\langle / \rangle$ not depend on the choice of submanifold map ϕ. This is the "conserved current" condition that is familiar from the standard treatises in quantum mechanics. Let us look at the Schrödinger equation from this point of view.

6. THE SCHRÖDINGER EQUATION FROM THE FIRST-ORDER POINT OF VIEW

Let X be space-time, i.e., R^4 with Cartesian coordinates

$$(x^\mu), \qquad 0 \leqslant \mu \leqslant 3$$

$$x^0 \equiv t, \qquad \text{the space components } x = (x^j), \qquad 1 \leqslant j \leqslant 3$$

Let ∂_0, ∂_i be the corresponding vector fields.

Let E be the product bundle

$$X \times \mathbf{C}$$

The cross sections of E are then the space of maps $\psi\colon X \to \mathbf{C}$, i.e., the complex-valued functions

$$\psi(x, t)$$

of space-time, i.e., the "Schrödinger wave functions." Let

$$\Delta = -i\hbar\partial_0 + (i\hbar\partial_j - a_j)(i\hbar\partial_j - a_j) + V \tag{13}$$

be the Schrödinger operator. (a_j, V are fixed functions of x and t. $\hbar$ is constant.) In order to convert this into a first-order operator, consider the inhomogeneous equation

$$\Delta\psi = f \tag{14}$$

Put

$$e_0 = \psi$$

$$e_j = (i\hbar\partial_1 - a_j)\psi$$

Then,

$$(i\hbar\partial_j - a_j)(e_j) = i\hbar\partial_0(e_0) - Ve_0 + f$$

Thus, (14) is equivalent to the following first-order system:

$$(i\hbar\partial_j - a_j)e_0 - e_j = 0 \qquad j = 1, 2, 3 \tag{15}$$

$$(i\hbar\partial_j - a_j)(e_j) + Ve_0 - i\hbar\partial_0(e_0) = f \tag{16}$$

Let us write this in matrix form. Set

$$\gamma = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{17}$$

$$p_j = i\hbar\partial_j \tag{18}$$

Then, (15) and (16) are equivalent to

$$\begin{pmatrix} V - p_0, & p_1 - a_1, & p_2 - a_2, & p_3 - a_2 \\ p_1 - a_1, & -1, & 0, & 0, \\ p_2 - a_2, & 0, & -1, & 0, \\ p_3 - a_3, & 0, & 0, & -1, \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

or

$$A(x, p)\gamma = \delta \tag{20}$$

Computing the co-Riemannian metric associated with this operator amounts to computing the

$$\mathrm{tr}(A^2)$$

Now, using (19)

$$\mathrm{tr}(A^2) = (V - p_0)^2 + (p_1 - a_1)^2 + (p_2 - a_2)^2 + (p_3 - a_3)^2 + (p_1 - a_1)^2 + 1 + (p_2 - a_2)^2 + 1 + (p_3 - a_3)^2 \tag{21}$$

We see that the co-Riemannian metric—essentially the quadratic terms in the p's in (21)—is just the flat (positive) metric on R^4.

One might ask next how the Galilean group affects all this. In Hermann (1970b) it is shown that the Galilean group (considered as a transformation group on R^4) acts as "conformal" symmetry of the Schrödinger operator. Let us now return to the general situation to investigate this point.

7. CONFORMAL ACTION ON FIRST-ORDER LINEAR DIFFERENTIAL OPERATORS AND THEIR SYMBOLS

Let us return to the case of a complex vector bundle E over a manifold X. Let $\pi: E \to X$ denote the bundle projection map. For $x \in X$, let

$$E(x) = \pi^{-1}(x)$$

be the fiber of π, a finite-dimensional complex vector space.

Let G be a transformation group that acts on E as a vector bundle automorphism. Thus, each $g \in G$ is a diffeomorphism of E such that:

a. There is a diffeomorphism g_X on X such that

$$\pi g = g_X \pi$$

G thus induces a group G_X of transformations on the base such that the map $g \to g_X$ of $G \to G$ is a homomorphism. For $x \in X$, each $g \in G$ maps the fiber $E(x)$ onto the fiber $E(gx)$.

b. Each $g \in G$ maps the fiber vectors spaces $E(x)$ *linearly* and *isomorphically* onto $E(gx)$.

Such a geometric action of G on E induces a *linear* action of G on the vector space $\Gamma(E)$ of cross sections

$$g(\gamma)(x) = g^{-1}(\gamma(g(x))) \qquad \text{for } g \in G, x \in X \tag{22}$$

This construction of linear representations of groups is basic in physics (e.g., in quantum field theory) and in mathematics ("induced representations").

A special sort of bundle automorphisms are the (scalar) *gauge transformations*. An element of this group is a mapping

$$m\colon X \to \mathbb{C} - (0)$$

Such an element acts on E and on $\Gamma(E)$ as follows:

$$me = m(x)e \qquad \text{for } x \in X, e \in E(x) \tag{23}$$

$$(m\gamma)(x) = m(x)\gamma(x) \qquad \text{for } x \in X, \gamma \in \Gamma(E) \tag{24}$$

Let $\Gamma(E)$ be the group of such transformations.

Now, let G be a given group of linear bundle automorphisms, and let

$$D, D'\colon \Gamma(E) \to \Gamma(E)$$

be linear differential operators. We say that $g \in G$ *transforms* D *into* D' if the action of g on $\Gamma(E)$ transforms D' into D, i.e., if

$$g^{-1}D'g = D \tag{25}$$

or

$$D'g = gD \tag{26}$$

Now, relation (26) implies the following property:

$$\text{If } \gamma \in \Gamma(E) \text{ is a solution of } D(\gamma) = 0,$$

$$\text{then } \gamma' = g(\gamma) \text{ is a solution of } D'(\gamma') = 0 \tag{27}$$

Proof. Equation (26) means that

$$\begin{aligned} D'(g\gamma) &= gD(\gamma) \\ &= 0 \qquad \text{if } D(\gamma) = 0 \end{aligned}$$

Thus, if $D = D'$, (27) means that *g is a symmetry of the differential equation* $D = 0$, i.e., it maps one solution into another. There are more general possibilities for generating such symmetries, which we shall now describe.

Suppose $g \in G$, D, D' and $m \in M(E)$ are such that

$$g^{-1} D' g = mD \tag{28}$$

Definition. If (28) is satisfied, we say that the pair $(g, m) \in G \times M$ *conformally* maps D onto D'. Again, if (28) is satisfied then, for each solution γ of $D = 0$, $g(\gamma)$ is a solution of $D' = 0$.

> *Theorem 7.1.* Let (g, m) be a conformal symmetry of the differential operator D. Let g_X be the diffeomorphism of the base space X associated with the bundle automorphism g: $E \to E$. Let
>
> $$\rho \colon T^d(X) \times T^d(X) \to \mathbb{C}$$
>
> be the co-Riemannian metric associated with D. Then g_X acts as a conformal transformation on the co-Riemannian metric, i.e.,
>
> $$(g_X)_*(\rho) = f\rho \qquad \text{for some } f \in \mathfrak{F}(X) \tag{29}$$

The proof follows routinely from the definitions, and is left to the reader.

This result is useful because it can serve to determine the structure of the group G_D of conformal symmetries of the operator D. The map $g \to g_X$ is a *homomorphism* of G_D onto a subgroup of the group of conformal automorphisms of ρ. If ρ is nondegenerate, the latter is a group about which there is much known. [For example, if ρ is the Lorentz metric $X = R^4$, then the group of conformal isometries is $O(4,2)$, which is in turn the basic group underlying Penrose's "Twistors," and other mass-zero Poincaré-invariant theories.] Of course, to complete the description of the algebraic structure of G_D, it is necessary to know the kernel of the homomorphism $g \to g_X$, to know the conformal symmetries of D, which acts as (non-Abelian) "pure" *gauge* transformations, i.e., preserve each fiber of the vector bundle $E \to X$. Let us briefly consider the conditions for this in terms of a local coordinate system (x^μ), $0 \leqslant \mu, \nu \leqslant 3$, for X and a product structure $E = X \times \mathbb{C}^n$ for E. D is then of the form

$$D = A^\mu \partial_\mu + A$$

where (A^μ, A) are 4×4 matrices that depend on x. Then, $\Gamma(E)$ consists of the maps γ: $X \to \mathbb{C}^n$. If g is a "pure" gauge transformation, it is of the form

$$(g\gamma)(x) = M(x)\gamma(x)$$

where $x \to M(x)$ is a map $X \to GL(n,\mathbb{C})$. Hence, the condition that such a g be a conformal symmetry is

$$A^\mu \partial_\mu M + AM = fM(A^\mu \partial_\mu + A) \qquad \text{for some } f \in \mathfrak{F}(X)$$

or

$$A^\mu M = fMA^\mu \tag{30}$$

$$A^\mu \partial_\mu (M) + AM = fMA \tag{31}$$

The compatibility of equations (30) and (31) is thus a necessary condition for the existence of conformal gauge symmetry of the differential equations.

8. THE PROJECTIVE SPACE OF A HERMITIAN SPACE AS THE QUANTUM STATE SPACE

We now leave the study of linear differential operators in order to discuss certain relations with the more conventional Hilbert space formulation of quantum mechanics. We shall work with what the functional analysts call "incomplete Hilbert spaces." To avoid this awkward term, we call them *Hermitian spaces*, which we now define.

Let H be a complex vector space; denote a typical element by ψ. To define H as a *Hermitian space* means to define an R-bilinear map

$$(\psi_1, \psi_2) \to \langle \psi_1 / \psi_2 \rangle$$

$$H \times H \to \mathbb{C}$$

such that

$$\langle \psi_1 / \psi_2 \rangle = \langle \psi_2 / \psi_2 \rangle^*$$

$$\langle \psi_1 / c\psi_2 \rangle = c\langle \psi_1 / \psi_2 \rangle$$

$$\langle \psi / \psi \rangle \geqslant 0$$

equality holds only if $\psi = 0$

If A: $H \to H$ is a linear map, the *adjoint* (if it exists), is a linear map A^* such that

$$\langle \psi_1 / A\psi_2 \rangle = \langle A^* \psi_1 / \psi_2 \rangle$$

The *projective space* $P(H)$ is the space whose elements are the one-dimensional linear subspaces σ of H. The *tangent bundle* $T(P(H))$ is the space of ordered pairs:

$$(\sigma, v)$$

$$\sigma \in P(H), \qquad v \in H/\sigma$$

The map $T(P(H)) \to P(H)$,

$$(\sigma, v) \to \sigma$$

defines $T(P(H))$ as a complex vector bundle over $P(H)$.

If H has a Hermitian structure, a Hermitian structure is induced on each of the fibers of the vector bundle $T(P(H))$. For $\sigma \in P(H)$, i.e., σ a one-dimensional linear subspace of H, let $\sigma^{\perp}$ be the orthogonal complement of σ in H. Since σ is finite dimensional, H is a direct sum of σ and $\sigma^{\perp}$. (Note the classical argument works in this "incomplete" case, since σ is finite dimensional.) Thus, H/σ can be identified with $\sigma^{\perp}$. The tangent bundle to the projective space can thus be identified with $\sigma^{\perp}$, a subspace of H.

Now, define a *symplectic structure* for $\Gamma(H)$, i.e., a real-valued, nondegenerate skew-symmetric two-form ω on $T(P(H))$ (it is readily verified that its exterior derivative is zero):

$$\omega(\psi_1, \psi_2) = i(\langle \psi_1 / \psi_2 \rangle - \langle \psi_2 / \psi_1 \rangle) \tag{32}$$

This form can be used to define a *Poisson bracket* $\{,\}$ for real-valued functions on $P(H)$.

Each Hermitian symmetric linear map $A: H \to H$ defines a real-valued function f_A on $P(H)$:

$$f_A(\sigma) = \frac{\langle \psi / A\psi \rangle}{\langle \psi / \psi \rangle} \tag{33}$$

where ψ is an element that generates ψ. The following results are proved in Hermann (1973b).

Theorem 8.1. If A_1, A_2, A_3 are Hermitian linear maps with

$$A_3 = i[A_1, A_2]$$

Then

$$f_{A_3} = \{f_{A_1}, f_{A_2}\}$$

Theorem 8.2. Suppose that H is a finite-dimensional vector space of dimension n. Let ω^n be the differential form of degree $2n$ on $P(H)$, i.e., the volume element form associated with the symplectic structure. Then,

$$\int_{P(H)} f_A \omega^n = \mathrm{tr}(A)$$

This relation is a key link between "classical" and "quantum" statistical mechanics. It relates the symplectic geometry of $P(H)$ (hence "classical mechanics") to a key invariant of the Hermitian structure. Here is another such link.

Let A: $H \to H$ be a Hermitian operator. Consider the Schrödinger equation:

$$i\hbar \frac{d\psi}{dT} = A\psi \tag{34}$$

to be solved for curves $t \to \psi(t)$ in H. We now show how these equations are the Hamilton equations associated with the symplectic structure on $P(H)$.

Consider a solution $t \to \psi(t)$ of (34) such that $\psi(t) \neq 0$ for all t. Let $\sigma(t)$ be the projection on $P(H)$. Let f_A be the real-valued function A defined on $P(H)$ [Formula (33)].

Theorem 8.3. The curve $t \to \sigma(t)$ in $P(H)$ that the solution $t \to \psi(t)$ of (34) defines, is then a solution of Hamilton's equations (with respect to the symplectic structure) with Hamiltonian $f_{A/\hbar}$.

Again, the proof of this is a routine chasing of definitions.

9. THE PROJECTIVE SPACE ASSOCIATED WITH SCHRÖDINGER PARTICLE MECHANICS AND THE COTANGENT BUNDLE OF A SPACE OF MEASURES ON THE CLASSICAL CONFIGURATION SPACE

We have seen that the natural symplectic manifold for abstract quantum mechanical dynamical systems is the projective space of a Hermitian space H. Let us now specialize H to be the Hermitian space appropriate to a particle moving on the real line, $-\infty < x < \infty$, denoted X. Let H be the space of complex valued, C^∞, rapidly decreasing functions

$$x \to \psi(x)$$

on X.

$$\langle \psi_1 / \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1(x)^* \psi_2(x)\, dx \tag{35}$$

Let $S(H)$ be the unit sphere in H, i.e., the set of $\psi \in H$ such that

$$\langle \psi / \psi \rangle = 1 \tag{36}$$

Let $U(1)$ be the multiplicative group of complex numbers of absolute value one. $U(1)$ acts freely on $S(H)$.

$$(\lambda, \psi) \to \lambda\psi \tag{37}$$

$$\lambda \in U(1), \qquad \psi \in S(H)$$

$P(H)$ is the orbit space of the action of $U(1)$ on $S(H)$.

Given a $\psi \in S(H)$ and a real number $\hbar > 0$ (to be identified with Planck's constant), it can be written in "polar" form:

$$\psi = \sqrt{P}\, e^{iS/\hbar} \tag{38}$$

where $x \to P(x)$, $S(x)$ are rapidly decreasing, real-valued functions such that

$$P(x) \geqslant 0 \tag{39}$$

$$\int_{-\infty}^{\infty} P(x)\, dx = 1 \tag{40}$$

The role of the decomposition (38) in the description of the "classical limit" of quantum mechanics (WKB, etc.) is well known. It is also one of the sources in physics of the modern theory of "Fourier integral operators" (Treves, 1980). (Another source is the theory of classical wave propagation.)

Let $\mathcal{P}(X)$ be the space of probability measures on X that are defined by everywhere nonzero, one-differential forms $P\,dx$ on X with $x \to P(x)$ a C^∞ rapidly decreasing function satisfying

$$P(x) > 0 \qquad \text{for } x \in X \tag{41}$$

$$\int_{-\infty}^{\infty} P(x)\,dx = 1 \tag{42}$$

The *tangent bundle* $T(\mathcal{P}(X))$ is the set of pairs $(P\,dx, \theta)$, with

$$P\,dx \in \mathcal{P}(X) \tag{43}$$

θ is a smooth, rapidly decreasing one-form on X such that

$$\int_X \theta = 0 \tag{44}$$

$T(\mathcal{P}(X))$ is a vector bundle over $\mathcal{P}(X)$; the fiber is the form θ satisfying (44). The *dual bundle* (which we think of as the cotangent bundle to the space of smooth probability measures on X) is constructed as follows. Let $\mathcal{F}(X)$ denote the space of C^∞, real-valued functions on X. Let the additive group R of real numbers act on $\mathcal{P}(X) \times \mathcal{F}(X)$ as follows:

$$(c, (P\,dx, S)) \to (P\,dx, (S + c)) \tag{45}$$

Then $T^d(\mathcal{P}(X))$ is defined as the orbit space of $T(\mathcal{P}(X))$ under this action. The duality between the fibers of $T^d(\mathcal{P}(X))$ and $T(\mathcal{P}(X))$ is defined as follows:

$$\langle (\mathcal{P}\,dx, \theta), (\mathcal{P}\,dx, S) \rangle = \int_X S\theta \tag{46}$$

Notice that the condition (42) implies that this functional is invariant under the group action (43).

Theorem 9.1. For each $\hbar > 0$, the space $T^d(\mathcal{P}(X))$ can be mapped into $P(H)$, where H is the "Schrödinger" Hermitian space. Namely, define

$$\phi_\hbar(P\,dx, S) = \text{image of } \sqrt{P}\, e^{iS/\hbar} \text{ in } P(H) \tag{47}$$

ϕ pulls back the natural symplectic form on $SP(H)$ to the natural symplectic form on cotangent bundles.

Proof. First, notice that ϕ_h is indeed well defined by formula (47). For if $(P\,dx, S)$, $(P'\,dx, S')$ are on the same orbit of the action [via (46)] of R, then the right side of (47) is equal.

A tangent vector v to $T^d(\mathcal{P}(X))$ at the point $(P\,dx, S)$ is a pair $(Q(x)\,dx, S)$, where $Q(x)\,dx$ is a one-form such that

$$\int Q(x)\,dx = 0$$

and S is a real-valued, C^∞, rapidly decreasing function. The symplectic form ω is defined as follows:

$$\omega((Q\,dx, f),(Q'\,dx, f')) = \int_X (f'Q\,dx - fQ'\,dx) \tag{48}$$

The proof of the rest of Theorem 9.1 is a routine verification of formulas that is left to the reader.

In Hermann (1965) one will find further work, particularly showing how "observables," i.e., real-valued functions on states, may be defined, and their relation in the "classical" and "quantum" cases.

10. EMBEDDING OF THE CLASSICAL MECHANICAL SYMPLECTIC STATE MANIFOLD INTO THE QUANTUM MECHANICAL ONE AND THE "HYDRODYNAMICAL" INTERPRETATION OF QUANTUM MECHANICS

Given a (finite-dimensional) manifold X that is the configuration space of a classical mechanical system (with a finite number of degrees of freedom), we have constructed the space $\mathcal{P}(X)$ of *smooth* probability measures on X, whose cotangent bundle may be identified with the projective space associated with the Schrödinger space. One may ask for the relation between these two symplectic "manifolds," one finite, the other infinite dimensional. (Unfortunately, the space of measures, even the "smooth" ones, admits nothing like any of the infinite-dimensional manifold structures that have been defined rigorously in the functional analysis–global analysis literature.)

However, it was shown in Hermann (1965) that, if one adds to $T^d(\mathcal{P}(X))$ certain "ideal" elements, there was a natural embedding of $T^d(X)$ into $T^d(\mathcal{P}(X))$. The classical dynamics is a one-parameter flow on $T^d(X)$ that preserves the simplectic structure. The quantum dynamics is a flow on $T^d(\mathcal{P}(X))$ with Planck's constant h appearing as a parameter.

(This already suggests that the deformation of groups and geometric structures plays a role!) Thus, we can envisage flows in $T^d(\mathcal{P}(X))$ that leave invariant precisely, or to a certain order, the space corresponding to the embedding of $T^d(X)$ into $T^d(\mathcal{P}(X))$. In essence, this was already done in the 1920s by the physicists, under the name "the hydrodynamical interpretation of quantum mechanics" (Madelung, 1926).

To briefly sketch this embedding, let us restrict attention to the simplest case, a particle on the line X with coordinate $x \in R$. As usual (Abraham & Marsden, 1978), identify $T^d(X)$ with R^2, with coordinates labeled

$$(x, y)$$

(y is the momentum.) $\mathcal{P}(X)$ is identified with the differential forms

$$p(x)\,dx$$

$$p(x) \geqslant 0$$

$$\int p(x)\,dx = 1$$

However, now allow distributions as coefficients of these differential forms; for example, elements of the form

$$\delta(x-a)\,dx$$

where $x \to \delta(x)$ is the Dirac delta function.

Now, associate with a point $(x_0, y_0) \in T^d(X)$ the following element $(p(x)\,dx, S(x))$, of $T^d(\mathcal{P}(X))$:

$$\begin{aligned} p(x) &= \delta(x-x_0)\,dx \\ S(x) &= xy_0 \end{aligned} \tag{49}$$

This defines a mapping

$$T^d(X) \to T^d(\mathcal{P}(X))$$

which has certain natural properties. Further details may be found in Hermann (1965) and volume 2 of Hermann (1970b).

11. CLOSING REMARKS

As mathematicians and physicists have come to realize (again!), differential geometry and physics have many common interests. Geometers have developed a warehouse of "geometric" structures and concepts, and physicists have the contact with reality that is needed to keep the study of such mathematical generalities from sterility.

In the period 1925–1930 the physicists (with Dirac at the lead) introduced a wealth of new ideas, many with a strong underlying geometric component, whose mathematical structure has not yet been fully explored. (After all, mathematicians lived on the Newtonian ideas for two hundred years!)

In this paper, I have suggested two such areas: the geometric structure of linear differential operators [which also ties in with recent developments in analysis (Treves, 1980; Taylor, 1981)], and the theory of infinite-dimensional symplectic manifolds.

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